

Calculus III: Coursework 7
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The deadline is 5PM on Tuesday November 21st. Hand-in is to the green box in the basement of the Maths building.

1. Find, to the nearest degree, the Earth polar coordinates of the following cities:
 - (a) Antananarivo also known as Tananarive (Madagascar)
 - (b) Chihuahua (Mexico)
 - (c) Makkah or Mecca (Saudi Arabia)

[10]

Answer

Upon looking up latitudes and longitudes:

- (a) Antananarivo is at 19S 48E hence $\theta = 109^\circ$, $\phi = 48^\circ$.
- (b) Chihuahua is at 29N 106W, so $\theta = 61^\circ$, $\phi = 254^\circ$
- (c) Makkah is at 21N 40E, so $\theta = 69^\circ$, $\phi = 40^\circ$.

Marks: Having the concept clear (4), lookups and calculation (3×2) [Marker: allow 47E for Antananarivo, 28N for Chihuahua and 39E for Mecca; I do not want to penalize students for failing to remember that there are 60 minutes in a degree rather than 100. The more detailed figures are respectively 47°31'E, 28°38'N and 39°49'E respectively.]

2. In the accompanying figures the vector $\mathbf{e}_1$ is recognizable as $\mathbf{e}_z$ since it points towards increasing z .

Figure 1: Figures for question 2

Identify the other basis vectors $\mathbf{e}_2, \dots, \mathbf{e}_6$, including minus signs if needed. Justify your identifications. [10]

Answer

On the left we have cylindrical coordinates with $\mathbf{e}_2 = \mathbf{e}_\rho$ and $\mathbf{e}_3 = \mathbf{e}_\phi$.

On the right we have spherical polar coordinates with $\mathbf{e}_4 = -\mathbf{e}_\theta$ (since it points towards decreasing θ), $\mathbf{e}_5 = \mathbf{e}_r$, and $\mathbf{e}_6 = \mathbf{e}_\phi$.

Marks: Correct identification $5 \times$ (1 for sign+ 1 for direction), with some suitable explanation

3. Show that about 4% of the Earth's surface lies north of the Arctic circle.

Remark: You can make the final numerical estimate without using a calculator; to do so, remember Taylor series. [13]

Answer

In Earth polar coordinates, the Arctic circle is at $\theta_1 \simeq 24^\circ$.

The area from $\theta = 0$ to $\theta = \theta_0$ is

$$R^2 \int_0^{2\pi} d\phi \int_0^{\theta_0} \sin \theta d\theta$$

where R is the Earth's radius.

Integrating gives

$$2\pi R^2(1 - \cos \theta_0) = 4\pi R^2 \sin^2 \frac{1}{2}\theta_0$$

For the area of the whole Earth, $\theta_0 = \pi$ giving $4\pi R^2$. For the Arctic circle, $\theta_0 = \theta_1$, so the fraction north of the Arctic circle is $\sin^2 \frac{1}{2}\theta_1 \simeq \sin^2 12^\circ$.

Since 12° is a small angle, we can apply $\sin^2 x \simeq x^2$. We have $12^\circ = \frac{1}{15}\pi$ radians $\simeq 0.2$, resulting in 4%.

Marks: Setting up the integral correctly (5), getting to $\sin^2 \frac{1}{2}\theta_1$ or equivalent (5), correct numerical approximation (3) (Students may use calculators for the numerical part).

4. Invert the relation

$$\begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}$$

for cylindrical coordinates, to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in terms of $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z$.

Hence show that the position vector $\mathbf{r}$ can be expressed as $\rho\mathbf{e}_\rho + z\mathbf{e}_z$. [12]

Answer The given matrix is orthogonal. Hence

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix}$$

Computing $\mathbf{r} = \rho \cos \phi \mathbf{i} + \rho \sin \phi \mathbf{j} + z\mathbf{k}$ gives the result.

Marks: Inverse (6) (Students may directly compute the inverse without realising it follows from orthogonality: do not penalise correct but long methods), position vector (6).

5. Show that in cylindrical polar coordinates

$$\dot{\mathbf{e}}_\rho = \dot{\phi} \mathbf{e}_\phi \quad \dot{\mathbf{e}}_\phi = -\dot{\phi} \mathbf{e}_\rho$$

and hence show that

$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\mathbf{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z.$$

[12]

Answer

Since

$$\mathbf{e}_\rho = \cos \phi \mathbf{i} + \sin \phi \mathbf{j} \quad \mathbf{e}_\phi = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}$$

the first result follows on differentiating.

We also know (from the previous question)

$$\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z.$$

Differentiating, and substituting the first result we get

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z.$$

(Here $\dot{\mathbf{e}}_z = \mathbf{0}$.) Doing likewise again we get the second result.

Marks: First part (4), velocity (4), acceleration (4).

6. Consider the two-dimensional coordinate system (α, β) defined by

$$x = \sinh \alpha \sin \beta, \quad y = \cosh \alpha \cos \beta.$$

Curves of constant α are ellipses, while curves of constant β are hyperbolae.

Show that this coordinate system is orthogonal.

Hint: Can you show that $\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = 0$?

[13]

Answer

Orthogonality means $\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = 0$.

Since by definition

$$\mathbf{e}_\alpha = \frac{\frac{\partial \mathbf{r}}{\partial \alpha}}{\left| \frac{\partial \mathbf{r}}{\partial \alpha} \right|} \quad \mathbf{e}_\beta = \frac{\frac{\partial \mathbf{r}}{\partial \beta}}{\left| \frac{\partial \mathbf{r}}{\partial \beta} \right|}$$

it is enough to show that

$$\frac{\partial \mathbf{r}}{\partial \alpha} \cdot \frac{\partial \mathbf{r}}{\partial \beta} = 0.$$

Starting from

$$\mathbf{r} = \sinh \alpha \sin \beta \mathbf{i} + \cosh \alpha \cos \beta \mathbf{j}$$

we have

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \alpha} &= \cosh \alpha \sin \beta \mathbf{i} + \sinh \alpha \cos \beta \mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial \beta} &= \sinh \alpha \cos \beta \mathbf{i} - \cosh \alpha \sin \beta \mathbf{j}\end{aligned}$$

whose dot product works out to be zero.

Marks: Some clue about what $\mathbf{e}_\alpha, \mathbf{e}_\beta$ are (4), correct calculation of $\mathbf{e}_\alpha, \mathbf{e}_\beta$ (5), understanding about orthogonality (4).

7. Using appropriate polar coordinates, calculate

- (a) $\oint \mathbf{F} \cdot d\mathbf{r}$ around the circle $x^2 + y^2 = 1$ in the x, y plane.
- (b) $\int \mathbf{G} \cdot d\mathbf{S}$ over the disc whose boundary is the above circle.
- (c) $\int \mathbf{G} \cdot d\mathbf{S}$ over the unit hemisphere $r = 1, z > 0$.

where $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and $\mathbf{G} = \nabla \times \mathbf{F} = 2\mathbf{k}$.

What do you notice about the answers?

[30]

Answer (a) Use plane polars (here called (ρ, ϕ)). Express the path as

$$x = \cos \phi, \quad y = \sin \phi, \quad z = 0, \quad 0 \leq \phi \leq 2\pi$$

hence

$$d\mathbf{r} = (-\sin \phi \mathbf{i} + \cos \phi \mathbf{j}) d\phi$$

We have

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (\sin^2 \phi + \cos^2 \phi) d\phi = 2\pi.$$

Marks: Writing path correctly (4), understanding what $\mathbf{F} \cdot d\mathbf{r}$ means (3), simplification (3).

(b) Use cylindrical polars, so for the disc, $d\mathbf{S} = \rho d\rho d\phi \mathbf{k}$ and we get

$$2 \int_0^1 \rho d\rho \int_0^{2\pi} d\phi = 2\pi.$$

Marks: correct integrand over disc (5), integration over disc (4).

(c) For the hemisphere $r = 1$, $d\mathbf{S} = \sin \theta d\theta d\phi \mathbf{n}$ where $\mathbf{n}$ is the unit normal $\mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{e}_r$, so $\mathbf{k} \cdot \mathbf{n} = z = \cos \theta$, and we have

$$2 \int_0^{\pi/2} \cos \theta \sin \theta d\theta \int_0^{2\pi} d\phi = 2\pi.$$

Marks: correct integrand over hemisphere (5), integration over hemisphere (4).

Noting the answers all agree: 2 marks.